Are Crude Oil and Natural Gas extreme prices interdependent?

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Abstract

We investigated the relationship between extreme prices of crude oil and natural gas. We found evidence that extreme events in these markets are interdependent and exhibit a self-excited dynamic behavior, where the intensity of extreme events in the oil market influences the occurrence rate and magnitude of extreme events in the natural gas market. In addition, we have determined that it is possible to better manage the Value-at-Risk, when both markets interact.

***Keywords*:** Interdependence, Hawkes-POT, Value-at-Risk, Extreme Value Theory

1 Introduction

With the advanced development of society and the growth of new technologies, energy consumption has become a research topic of great importance. Among the most commercialized energy commodity are oil and natural gas. Numerous studies have focused on analyzing the relationship between the prices of these markets, using commonly co-integration and error correction models, frameworks based on volatility and copula models. (Bildirici and Bakirtas, 2014; Hartley et al., 2008; Krichene, 2002; Efimova and Serletis, 2014; Ewing et al., 2003; Pindyck and Others, 2004; Aloui et al., 2014; Grégoire et al., 2008).

Recently some researchers have examined the behavior of these energy commodities focusing only in extreme prices, using POT models, Maximum blocks, Conditional-EVT models and Extreme copulas (Aloui et al. 2014; Ghorbel and Trabelsi 2014; Gülpinar and Katata 2013). The analysis of the extreme events allows to obtain more complete information about these markets in periods of financial crisis. Though, stylized facts presented in the financial

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series, cannot be captured by methodologies that assume independent and identically distributed data, as the models currently do.

Unlike previous studies, we propose a new specification for the multivariate extension of the Hawkes-POT univariate model (Chavez-Demoulin et al., 2005). This methodology based on self-excited point processes and extreme value theory, incorporates the internal history of past extreme events to capture behavior by cluster on extreme events. Besides, works only with extreme observations, avoiding that mean values biasing estimates.

We determine how the intensity of extreme events in the natural gas market is influenced by the dynamic of the occurrence and magnitude of extreme events in the oil market, and vice versa. Also, we analyze if it is possible better manage the value at risk (VaR) in these markets, if the way in which they interact is known. Additionally, to complement the information we incorporate estimates with positive extreme observations.

However, these methodologies assume independent and identically distributed data, which cannot capture the actual behavior of financial series that present stylized facts like extreme clusters. We present a conditional model to the history of past events, it can capture the dynamics of extreme event clusters in financial series, characterizing in a better way the behavior of these markets in the queues of the distribution.

The contribution of this paper is twofold, on the one hand, we present an empirical analysis that allows analyzing the relationship between oil prices and natural in periods of extreme loss, relaxing the assumption of iid data. On the other hand, we present a methodological proposal, introducing a new specification for the multivariate extension of the Hawkes POT model.

This paper is organized as follows: In Section 2, we introduce the proposed methodology. Section 3 includes the empirical analysis with the description of the models, estimation results and analysis of the risk measures. Finally, Section 4 presents the conclusions, study limitations and recommendations for future research.

2 Methodology

We define $\{X_t^m\}_{t\geq 1}$ as the negative returns of the oil ($m = 1$) and the natural gas (m =2) markets. In addition, we define a sufficiently high threshold $u^m > 0$, about which all the observations are considered extreme. Each of these events occurs at a time T_i^m and is accompanied by a mark, in our case the magnitude of the event above the threshold $Y_i^m = X_{T_i}^m - u^m$. Thus, a marked point process (MPP) can be obtained, $N(t) \equiv \sum \mathbb{1}_{\{T_i^m < t, y = Y_i^m\}}$, which corresponds to a stochastic process accounting for the number of extreme events up to time t and can depend on the history of the process $\mathcal{H}_t = \left(\{T_1, Y_1\} \ldots \{T_{\mathrm{N}(t)-1}, Y_{\mathrm{N}(t)-1}\}\right)$.

In terms of its conditional intensity, the MPP can be defined through two stochastic processes: a ground intensity $\lambda_g^m(t|\mathcal{H}_t)$, which determines the dynamic of the occurrence of extreme events, and a probability density function $g^m(y|\mathcal{H}_t, t)$, which characterizes the distribution of the marks

$$\lambda^m(t, y|\mathcal{H}_t) = \lambda_g^m(t|\mathcal{H}_t) g^m(y|\mathcal{H}_t, t). \tag{1}$$

The ground intensity follows a Hawkes structure

$$\lambda_g^m(t|\mathcal{H}_t) = \mu_m + \sum_{k=1}^{M} \eta_{mk} \cdot \sum\nolimits_{i:T_i^k<t} h_{mk}(y, t) \tag{2}$$

where μ_m is the baseline rate independent of the self-exciting process (exogenous intensity) and η_{mk} determines the influence of the component m through the intensity of events in the component k (endogenous process). In addition, an exponential kernel $h_{mk}(y, t) = \alpha_k \exp\left(\delta_{mk} y_i^k - \alpha_k\left(t - T_i^k\right)\right)$ is used to characterize the instant memory of the conditional intensity. In this expression, α_k is the rate of decay of the instant intensity and δ_{mk} is the parameter that captures the influence that the size of the extreme events k has on the intensity of the market m.

Following the central idea of the POT models, the marks follow a generalized Pareto distribution

$$g^m(y|\mathcal{H}_t, t) = \frac{1}{\beta^m(y|\mathcal{H}_t)}\left(1 + \xi^m \frac{y_m}{\beta^m(y|\mathcal{H}_t)}\right)^{-\frac{1}{\xi^m}-1} \tag{3}$$

where $\xi^m > 0$ denotes the shape parameter, while the scale parameter is defined as follows

$$\beta^m(y|\mathcal{H}_t) = \beta_0^m + \sum_{k=1}^{M} \beta_k^m \cdot \sum\nolimits_{t_{j=1}^k<t} h_{mk}(y, t) \tag{4}$$

and seeks to relate the instant intensity of the self-exciting process and the size of the extreme events. Finally, the estimation of the model is through the maximization of its log-likelihood

$$\ln\mathcal{L} = \sum_{m=1}^{M} \sum_{j=1}^{N^m(T)} \left\{\ln g\,(y|\mathcal{H}_t, t) + \ln \lambda_g^m\,(t|\mathcal{H}_t)\right\} - \sum_{m=1}^{M} \int_0^T \lambda_g^m(s|\mathcal{H}_s) ds. \tag{5}$$

The VaR in terms of a Hawkes-POT model corresponds to

$$VaR_{\alpha}^{t+1} = u + \frac{\beta(y|\mathcal{H}_t)}{\xi}\left(\left(\frac{1-\alpha}{\lambda_g(t|\mathcal{H}_t)}\right)^{-\xi} - 1\right) \tag{6}$$

where the sub-index has been eliminated to facilitate the exposure (Herrera and Schipp, 2013).

The likelihood ratio test of unconditional coverage (*LRuc*), the likelihood ratio test of independence (*LRind*) and the likelihood ratio test of conditional coverage (*LRcc*) are used to evaluate the VaR accuracy (Christoffersen, 1998), For further details regarding these three tests, see Anexo 1. All tests are approved with p-values greater than 0.05, and evaluated considering three levels of trust: 0.95, 0.99, and 0.999.

3 Empirical Analysis

The data used correspond to daily prices from WTI crude oil and Henry Hub natural gas were obtained from the Federal Reserve Economic Data. We consider the position of an investor and used the negative log-returns of these prices, see Figure 1. The in-sample period used for the estimation includes observations from January 7, 1997 to December 31, 2014, whereas observations from January 2, 2015 to December 31, 2015 are used for backtesting.

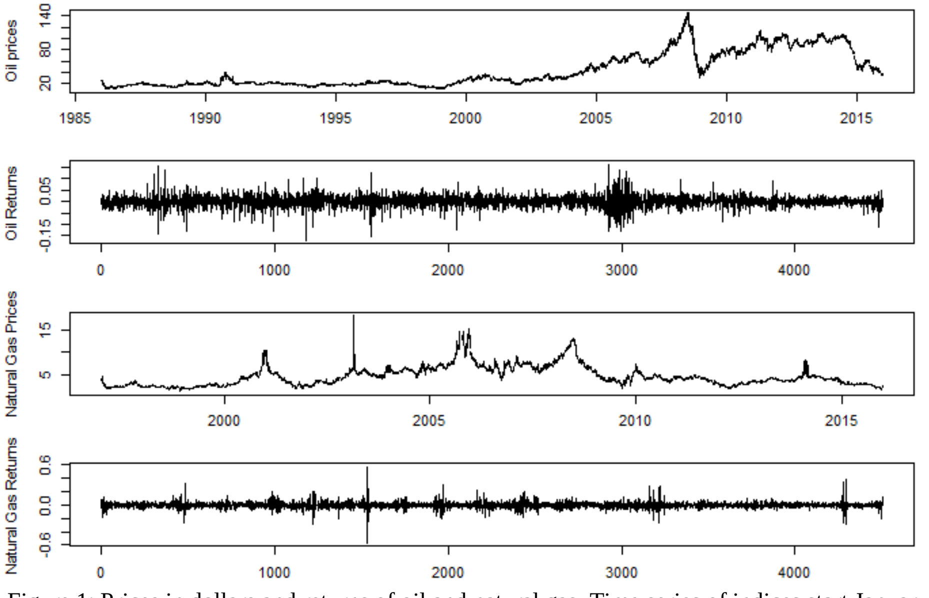

Figure 1: Prices in dollars and returns of oil and natural gas. Time series of indices start January 7, 1997 and end December 31, 2014

While, Table 1 shows statistical summary of data, this considers the most significant stylized factors in both returns of these commodities. Note that the returns show averages close to zero, with heavy tails and an excess of kurtosis. For the WTI market we also observed a clear negative skweness, indicating heavier queues for losses in this market. Surprisingly, natural gas shows a positive skewness. Finally both markets reject the null hypotheses of negative returns iid or normally distributed through the statistical tests Jarque Bera and Box-pierce, respectively.

	Oil WTI	*Natural Gas*
Mean	0,0002	0,0000
Standard deviation	0,0245	0,0453
Minimum	-0,1709	-0,5682
Maximum	0,1641	0,5767
Skewness	-0,1805	0,5946
Excess of Curtosis	5,0856	20,8041
Jarque Bera	0,0000	0,0000
Box-Pierce (5 lag)	0,0005	0,0000
Box-Pierce(10 lag)	0,0004	0,0000

Tabla 1:

3.1 Threshold selection

One of the most complex tasks in this area is to choose a threshold that allows to establish the minimum value in which all the observations will be considered extreme events. This selection plays an important role in the estimation of model parameters. The main difficulty is in the trade-off between bias and variance. If we choose a low threshold, the number of observations increases and the estimate becomes more accurate. However, with this choice we also introduce observations from the center and the estimation becomes biased.

There are different methods for choosing the threshold. Smith (1987) chooses to arbitrarily set a threshold of 90%, while Chavez-Demoulin and McGill (2012) recommend selecting thresholds between 92% and 95% through a graphical technique. Herrera (2013) proposes a methodology based on a sensitivity analysis of the VaR, choosing that threshold that is more stable to the estimates of this measure of risk. On the other hand, there are authors who use strategies based on Bayesian approaches (de Zea Bermudez and Amaral Turkman, 2003; de Zea Bermudez et al., 2001), minimization of mean square error (MSE), bootstrap techniques (Danielsson et al., 2001; Gomes and Oliveira, 2001) or graphical methods such as, mean residual life (or mean excess) plots, threshold stability plots, and adjustment diagnostics of a distribution such as probability plots, quantile plots and return level plots, among others. (Beirlant et al., 2004; Coles, 2001).

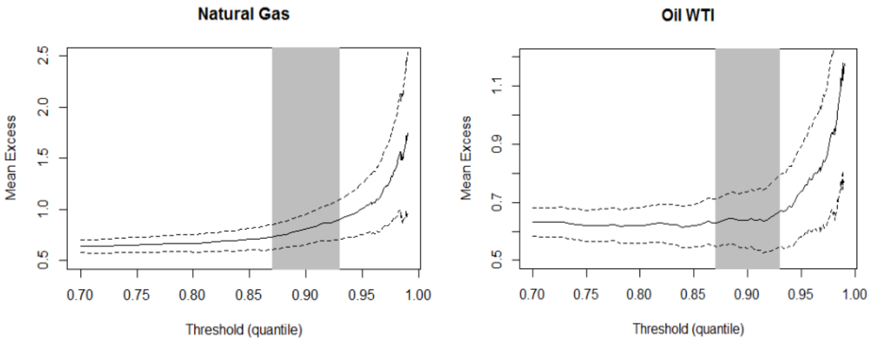

Figura 2: Residual life plot to select the threshold of events in each marginal (natural gas and WTI oil)

In this research, we follow the study proposed by Chavez-Demoulin and McGill (2012) and use the mean residual life plot technique introduced by Davison and Smith (1990). This technique consists of plotting the mean excess on the threshold in relation to different thresholds. According to this graph if the assumption that the excesses on the threshold have Generalized Pareto distribution, then the graph should be linear in u with slope ξ / (1-ξ). Figure 1 shows the results of this plot for the analyzed markets. A good compromise between bias and variance according to this graph would be a threshold between the 0.90 and 0.92 quantile for both financial series. We decided therefore to work with the maximum possible, 10% of the most extreme events for both series.

3.2 Interdependence and dynamic linkages

We estimated four Hawkes-POT models. Each of these models raises hypotheses of how extreme events interact. In Model 1 is a specification where both markets are completely related, allowing cross-excitement through their intensities as well as through the magnitudes of the extreme returns. On the opposite end is Model 2, which fully restricts the interaction between the intensities and magnitudes in both markets (i.e., $\eta_{12} = \eta_{21} = \beta_1^2 = \beta_2^1 = \delta_{12} = \delta_{21} = 0$), which would indicate independence in the intensity of occurrence and magnitudes of the extreme events in both markets. According to the results obtained for the two above models and in line with previous literature (Hartley et al., 2007), we proposed two unidirectional models to capture whether extreme events in the natural gas market are indeed affected by those occurring in the oil market. Model 3 considers that there can only be influence from the oil market to the natural gas market, i.e., there is no influence from the natural gas to the oil market through either its intensity, marks, or in the magnitude of the extreme events within of the self-exciting process (i.e., $\eta_{12} = \beta_2^1 = \delta_{12} = 0$). Finally, Model 4, in addition to including the restrictions of Model 3, limits the possibility that the intensity of the extreme events at the oil market has any influence on the magnitude of the extreme events occurring in the natural gas market ($\beta_1^2 = 0$).

			Model 1	Model 2	Model 3	Model 4
Intensity	Oil	μ_1	0.056	0.057	0.056	0.056
			(0.008)	(0.008)	(0.008)	(0.008)
		η_{11}	0.218	0.215	0.217	0.217
			(0.057)	(0.056)	(0.056)	(0.056)
		η_{12}	0.000			
			(0.001)			
		α_1	0.038	0.041	0.039	0.039
			(0.008)	(0.008)	(0.008)	(0.008)
		δ_{11}	0.633	0.638	0.637	0.637
			(0.071)	(0.069)	(0.070)	(0.070)
		δ_{12}	1.591			
			(1.765)			
	Natural Gas	μ_2	0.036	0.048	0.037	0.037
			(0.008)	(0.006)	(0.008)	(0.008)
		η_{22}	0.359	0.386	0.357	0.357
			(0.055)	(0.055)	(0.055)	(0.055)
		η_{21}	0.152		0.151	0.151
			(0.070)		(0.069)	(0.069)
		α_2	0.083	0.081	0.084	0.084
			(0.010)	(0.010)	(0.011)	(0.011)
		δ_{22}	0.307	0.302	0.308	0.308
			(0.026)	(0.025)	(0.026)	(0.026)
		δ_{21}	0.000		0.000	
			(0.002)		(0.002)	
Magnitude	Oil	β_0^1	0.349	0.353	0.351	0.351
			(0.044)	(0.044)	(0.044)	(0.044)
		β_1^1	1.236	1.222	1.233	1.233
			(0.261)	(0.255)	(0.258)	(0.258)
		β_2^1	0.000			
			(0.000)			
		ξ_{i1}	0.024	0.024	0.024	0.024
			(0.044)	(0.044)	(0.044)	(0.044)
	Natural Gas	β_0^2	0.196	0.194	0.197	0.197
			(0.030)	(0.030)	(0.030)	(0.030)
		β_2^2	2.340	2.369	2.333	2.333
			(0.265)	(0.266)	(0.265)	(0.265)
		β_1^2	0.000		0.000	
			(0.002)		(0.003)	
		ξ_{i2}	0.021	0.023	0.021	0.021
			(0.048)	(0.048)	(0.048)	(0.048)
		Log Lik	-3341.04	-3345.00	-3342.33	-3342.33
		AIC	6722.07	6718.00	6718.66	6714.66

Table 2: The fit of the parameters for the four models evaluated and their respective standard errors, as well as log-likelihood and Akaike information criterion.

Table 1 shows the results of the estimations. The best fit according to the AIC is Model 4; therefore, there is evidence that extreme events in both markets are interdependent, but unidirectionally from the oil to the natural gas market. In the case of Model 1, although it has a good fit, many of its interdependence parameters have values near zero. On the other hand, Model 2, considering that both markets are completely independent, is too restrictive to represent their behavior. Finally, although Model 3 can restrict the direction of interdependence, this does not fully capture the way in which the two markets interact.

The dynamics of conditional intensity for Model 4 is illustrated in Figure 1. The first two panels show the marks or magnitudes of the excesses above the threshold, and the two following panels show the intensity of the estimated conditional occurrence. Both markets show a simultaneous increase in the intensity and magnitude of the extreme events driven by episodes like the Asian Crisis and slightly during the Subprime Crisis (consistent with that described by Atil et al., 2014; Ramberg, 2010; Villar and Joutz, 2006). The barcode plot in the lower panel validates our results: light grey colors indicate independent events and dark colors simultaneous occurrences.

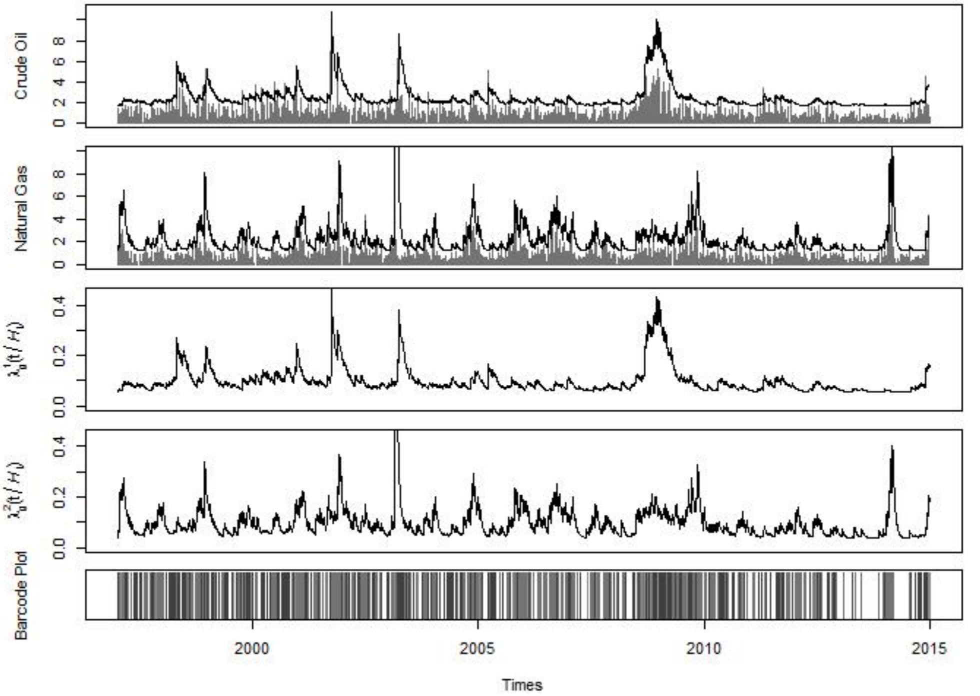

Figure 3: The first and second panel display the negative price returns of oil and natural gas and the estimations of the VaR at 99%. The third and fourth panel show the intensity of occurrence of these events. The last panel is a barcode illustrating the extreme events that occurred simultaneously.

Analyzing Model 4 in detail, in both markets the endogenous behavior of the extreme events (η_{11}, η_{22} and η_{21}) is higher than the exogenous one (μ_1 and μ_2). In particular, the rate of self-excitement is around 0.30 for both markets, whereas the rate of cross-excitement from the oil market towards the natural gas market is 0.15. A similar result is found for the positive influence of the magnitude of extreme events (δ_{11} and δ_{22}) on the intensity, as well as the intensity of the extreme events on their magnitude (β_1^1 and β_2^2). Finally, the response of a previous extreme event on the intensity function in both markets decays exponentially towards their baseline intensities, with a faster rate for natural gas than for the oil market(α_1 and α_2).

			2015				*2014-2015*			
			Excep	*LRuc*	*LRind*	*LRcc*	*Excep*	*LRuc*	*LRind*	*LRcc*
Model 1	Oil	0.95	18	0.13	0.10	0.08	24	0.83	0.12	0.29
		0.99	4	0.38	0.72	0.63	7	0.40	0.66	0.63
		0.999	0	0.48	1.00	0.78	3	**0.02**	0.85	0.06
	Natural Gas	0.95	19	0.08	0.19	0.09	28	0.55	**0.02**	0.05
		0.99	4	0.38	0.75	0.64	8	0.22	0.11	0.13
		0.999	3	**0.00**	0.83	**0.01**	1	0.54	0.95	0.82
Model 2	Oil	0.95	17	0.21	0.13	0.14	24	0.83	0.12	0.29
		0.99	4	0.38	0.72	0.63	8	0.22	0.61	0.41
		0.999	0	0.48	1.00	0.78	2	0.11	0.90	0.28
	Natural Gas	0.95	17	0.21	0.10	0.12	30	0.32	**0.01**	**0.02**
		0.99	4	0.38	0.75	0.64	9	0.11	0.14	0.09
		0.999	3	**0.00**	0.83	**0.01**	2	0.11	0.90	0.28
Model 3	Oil	0.95	17	0.21	0.13	0.14	23	0.67	0.14	0.30
		0.99	4	0.38	0.72	0.63	8	0.22	0.61	0.41
		0.999	0	0.48	1.00	0.78	2	0.11	0.90	0.28
	Natural Gas	0.95	19	0.08	0.19	0.09	28	0.55	**0.02**	0.05
		0.99	4	0.38	0.75	0.64	9	0.11	0.14	0.09
		0.999	3	**0.00**	0.83	**0.01**	1	0.54	0.95	0.82
Model 4	Oil	0.95	17	0.21	0.13	0.14	23	0.67	0.14	0.30
		0.99	4	0.38	0.72	0.63	8	0.22	0.61	0.41
		0.999	0	0.48	1.00	0.78	2	0.11	0.90	0.28
	Natural Gas	0.95	19	0.08	0.19	0.09	28	0.55	**0.02**	0.05
		0.99	4	0.38	0.75	0.64	9	0.11	0.14	0.09
		0.999	3	**0.00**	0.83	**0.01**	1	0.54	0.95	0.82

Table 3: Number of errors in the VaR adjustments and P value of the statistical tests LRuc, LRind and LRcc at 95%, 99% and 99.9%.

In a final exercise we determine if there is any advantage in the accuracy of the estimation of VaR forecasting, considering the synergies existing between the two markets. First, we investigated the fit of the VaR at different levels for one out-of-sample year (January 2, 2015 - January 31, 2015). The backtesting results are in Table 2. The four models showed only two exceptions to the 0.999 in natural gas in the test of conditional ($LRcc$) and non-conditional coverage ($LRuc$), respectively. From a statistical point of view, the tests do not seem to have enough power to reject the given sample sizes. Therefore, we decided to re-estimate the models, leaving the last two years as prediction period (January 2, 2014 - January 31, 2015), whereas the rest of the sample from January 7, 1997 is used for the estimation of the model.

In this case, unidirectional Models 3 and 4 show a better fit where only the independence hypothesis ($LRind$) was rejected in the exceptions of 0.95 VaR in the natural gas market. In Model 1 a rejection of the $LRuc$ test at VaR 0.999 in the oil market is also observed, whereas Model 2 presents a rejection in the $LRcc$ test at VaR 0.95 in the natural gas market

3.3 Positive extreme prices

Positive logarithmic returns were used to examine the interaction of these markets during periods of extreme gains. The analysis allows to evaluate if the relation between both energetic commodities exhibit a similar behavior in both tails of the distribution. The results of this analysis complement the information we obtained when analyzing negative extremes (section 3.2), providing tools that also allow us to evaluate investment strategies when these markets present important increases in their prices. Estimates were made using the 4 proposed models. The results are shown in Table A.1, exhibiting an asymmetric behavior in the tails. Unlike the analysis with negative ends, when studying positive ends the best fit is obtained with model 2, so it is not possible to establish a co-movement relationship when we analyze only positive extremes returns.

4 Conclusions

In this investigation we analyzed the degree of interdependence between the oil and natural gas markets from the point of view of the dynamics of negative extreme returns through a bivariate Hawkes-POT model. The estimates show that the interdependence is unidirectional, where the intensity of extreme events in the oil market influences the dynamics of the occurrence and magnitude of extreme events in the natural gas market. In these markets we observed an endogenous behavior of extreme events, responsible for the formation of clusters. The results from the VaR forecasting show that it is possible to improve the predictions if the way in which the two markets interact is known from the point of view of their co-movements at extreme levels. We present an analysis of the extreme positive returns, however in this context it was not possible to evidence the existence of co-movement between these markets.

5 Anexos

Positive Extreme Values

			Model 1	Model 2	Model 3	Model 4
Intensity	Oil	μ_1	0.054	0,058	0.056	0.056
			(0.012)	(0.011)	(0.012)	(0.012)
		η_{11}	0.242	0.217	0.229	0.232
			(0.086)	(0.081)	(0.085)	(0.085)
		η_{12}	0.000			
			(0.001)			
		α_1	0.029	0.035	0.032	0.032
			(0.010)	(0.011)	(0.011)	(0.011)
		δ_{11}	0.646	0.661	0.653	0.652
			(0.077)	(0.078)	(0.078)	(0.078)
		δ_{12}	2.096			
			NAN			
	Natural Gas	μ_2	0.039	0.049	0.039	0.039
			(0.009)	(0.006)	(0.009)	(0.009)
		η_{22}	0.366	0.378	0.366	0.367
			(0.056)	(0.057)	(0.056)	(0.057)
		η_{21}	0.111		0.105	0.106
			(0.076)		(0.074)	(0.074)
		α_2	0.078	0.076	0.079	0.078
			(0.011)	(0.011)	(0.011)	(0.011)
		δ_{22}	0.274	0.271	0.274	0.273
			(0.027)	(0.026)	(0.027)	(0.027)
		δ_{21}	0.000		0.000	
			(0.004)		(0.004)	
Magnitude	Oil	β_0^1	0.315	0.322	0.319	0.318
			(0.044)	(0.044)	(0.044)	(0.044)
		β_1^1	0.986	0.960	0.979	0.983
			(0.241)	(0.237)	(0.241)	(0.242)
		β_2^1	0.000			
			(0.001)			
		ξ_{i1}	0.040	0.035	0.039	0.040
			(0.052)	(0.051)	(0.052)	(0.052)
	Natural Gas	β_0^2	0.194	0.227	0.195	0.229
			(0,043)	(0.031)	(0.043)	(0.031)
		β_2^2	2.432	2.450	2.421	2.425
			(0.305)	(0.309)	(0.306)	(0.308)
		β_1^2	0.347		0.347	
			(0.347)		(0.342)	
		ξ_{i2}	0.023	0.028	0.025	0.029
			(0.048)	(0.049)	(0.048)	(0.049)
		Log Lik	-3342.51	-3345.44	-3343.78	-3344.47
		AIC	6725.02	6718.88	6721.56	6718.94

Table A.1: The fit of the parameters for the four models evaluated and their respective standard errors, as well as log-likelihood and Akaike information criterion.

1. likelihood ratio test

The first one is the likelihood ratio unconditional coverage test ($LRuc$), which aims to determine whether the number of errors obtained differs from the expected value, remaining consistent with the stated confidence level α in the risk quantification. Consider $H_t = \text{I}\,(r_t < -VaR_\alpha^t)$ as being the sequence of efficient predictions of the VaR; if $\text{E}[H_t] = \alpha$ with $H_t \overset{IID}{\sim} Bernoulli(\alpha)$, then under the null hypothesis $\text{H}_0\text{: E}[H_t] = \alpha$, for which the likelihood is defined as

$$lnL(\alpha;\ H_1, H_2, \dots, H_T) = (1-\alpha)^{n_0}\alpha^{n_1},$$

where n_0 is the number of correct predictions, while n_1 is the number of violations of VaR. On the other hand, the alternative hypothesis is defined as $\text{H}_1\text{: E}[H_t] \neq \alpha$

$$lnL(\hat{\pi};\ H_1, H_2, \dots, H_T) = (1-\pi)^{n_0}\pi^{n_1}$$

where $\hat{\pi} = \frac{n_1}{n_0+n_1}$ is the likelihood of π. Then, by means of a likelihood ratio test, one can test the unconditional coverage

$$LR_{uc} = 2[lnL(\hat{\pi};\ H_1, H_2, \dots, H_T) - lnL(\alpha;\ H_1, H_2, \dots, H_T)] \overset{asy}{\sim} \chi_1^2$$

with L being the likelihood of the binomial distribution and χ_1^2 the Chi-squared distribution with one degree of freedom. Whereas repeated mistakes result in significant losses for the investor, the likelihood ratio test of independence ($LRind$) verifies that there is no dependence over time between violations. Since H_t is a series of binary variables, we can model the dependence as a Markov chain whose first order transition matrix is defined by

$$\prod = \begin{bmatrix} 1-\pi_{01} & \pi_{01} \\ 1-\pi_{11} & \pi_{11} \end{bmatrix}$$

where $n_{ij} = \sum I\,(H_t = j\,|H_{t-1} = i)$ represents the number of transitions of state i to state j.

Under the null hypothesis, we observe that $\pi_{01} = \pi_{11} = \pi_0$, so the conditional likelihood at the first state has to be

$$L(\Pi_1;\ H_2, \dots, H_T|H_1) = (1 - \pi_{01})^{n_{00}+n_{10}} {\pi_{01}}^{n_{01}+n_{11}}$$

Under the alternative hypothesis, we have that $\hat{\pi}_{01} = \frac{n_{00}}{n_{00}+n_{01}}$ and $\hat{\pi}_{11} = \frac{n_{11}}{n_{10}+n_{11}}$ with likelihood

$$L(\Pi_2; H_2, \dots, H_T|H_1) = (1 - \pi_{01})^{n_{00}} {\pi_{01}}^{n_{01}} (1 - \pi_{11})^{n_{10}}\ {\pi_{11}}^{n_{11}}$$

Finally, the likelihood ratio test of independence for this statistic is defined by

$$LR_{uc} = 2[lnL(\hat{\pi};\ H_1, H_2, \dots, H_T) - lnL(\alpha;\ H_1, H_2, \dots, H_T)] \overset{asy}{\sim} \chi_1^2$$

which is asymptotically Chi-square distributed with one degree of freedom.

Providing a more global perspective, the likelihood ratio test of conditional coverage (LR_{cc}) simultaneously checks the previous two tests

$$LR_{cc} = LR_{uc} + LR_{ind},$$

where the likelihood ratio test is defined as

$$LR_{cc} = 2[lnL(\Pi_2;\ H_2, \dots, H_T|H_1) - lnL(\alpha;\ H_2, \dots, H_T|H_1)] \overset{asy}{\sim} \chi_{2,}^2$$

which is asymptotically Chi-squared distributed with two degrees of freedom.

6 References

Atil, A., Lahiani, A., Nguyen, D.K., 2014. Asymmetric and nonlinear pass-through of crude oil prices to gasoline and natural gas prices. Energy Policy 65, 567–573.

Brown, S.P. a., Yücel, M.K., 2008. What drives natural gas prices? Energy J. 29, 45.

Chavez-Demoulin, V., Davison, a. C., McNeil, a. J., 2005. Estimating value-at-risk: a point process approach. Quant. Financ. 5, 227–234.

Christoffersen, P., 1998. Evaluating interval forecasts. Int. Econ. Rev. (Philadelphia). 39, 841–862.

Davison, A.C., Smith, R.L., 1990. Models for Exceedances over High Thresholds. J. R. Stat. Soc. Ser. B 52, 1–4.

Hartley, P., Iii, K.B.M., Rosthal, J., 2007. The relationship between crude oil and natural gas prices. Inf. Adm. Off. Oil Gas 37.

Herrera, R., Schipp, B., 2013. Value at risk forecasts by extreme value models in a conditional duration framework. J. Empir. Financ. 23, 33–47.

Panagiotidis, T., Rutledge, E., 2004. Oil and gas markets in the UK: Evidence for from a cointegration approach. Energy Econ. 1–31.

Ramberg, D.J., 2010. The Relationship between Crude Oil and Natural Gas Spot Prices and its Stability Over Time by. Eng. Syst. Div.

Villar, J. a, Joutz, F.L., 2006. The relationship between crude oil and natural gas prices. Energy Inf. Adm. 1–43.

Yorucu, V., Bahramian, P., 2015. Price modelling of natural gas for the EU-12 countries: Evidence from panel cointegration. J. Nat. Gas Sci. Eng. 24, 464–472.